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Extremal Solutions of the Two-Dimensional *L*-Problem of Moments, II

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The extremal solutions of the truncated L-problem of moments in two real variables, with support included in a given compact set, are described as characteristic functions of semi-algebraic sets given by a single polynomial inequality. An exponential kernel, arising as the determinantal function of a naturally associated hyponormal operator with rank-one self-commutator, on the other hand provides a defining function for these semi-algebraic sets and, on the other hand, encodes in a closed form their moments. In order to understand the finite determination structure of the extremal sequences of moments we study analytic continuation properties of the corresponding exponential kernel and, separately, some cyclicity properties of the associated hyponormal operator. An intrinsic characterization of the exponential kernel is also discussed.

1. INTRODUCTION

In a previous paper [15], a special class of extremal solutions of the L-problem of moments in two variables (called degenerated solutions) was related via hyponormal operators to quadrature domains in the plane and to some rational functions associated to them. It is the aim of the present paper to apply the same techniques to all extremal solutions of the L-problem and to begin a study of the analytic and matricial objects arising from this investigation.

Let K be a compact subset of the complex plane, let L be a fixed positive constant, and let N be a fixed positive integer. We are interested in classifying and characterizing in intrinsic terms the moments

$$a_{nm} = \int_K \varphi(x, y)\, x^n y^m \, dA, \qquad 0 \leqslant m \leqslant m+n \leqslant N,$$

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of a measurable function ϕ on K which satisfies $0 \leqslant \phi \leqslant L$, dA-a.e. Above, and throughout the paper, dA stands for the planar Lebesgue measure. Note that because we are working with the two-dimensional Lebesgue measure, the alterations of null measure of the compact set K (such as adding or removing continuous exterior lines or internal slits) are not significant for our discussion.

Let Σ denote the collection of all vectors $a = (a_{kl})_{k+l \leqslant N} \in \mathbf{R}^d$ ($d = (N+1)(N+2)/2$) which arise as the moments of a function ϕ as before. It is clear that Σ is a compact convex subset of $\mathbf{R}^d$. Following Krein and his now classical convexity theory ([10, 8]) we infer that a point a belongs to the boundary of the set Σ if and only if it corresponds to the moments of a function ψ of the form

$$\psi = L\chi_{\Omega}, \qquad \Omega = \{(x, y) \in K; p(x, y) > 0\},$$

where p is an arbitrary polynomial of degree at most equal to N (with real coefficients) and χ_D is the characteristic function of the set D. Moreover, it is well known that only in this case does the above moment problem have a unique solution. (The role of the bound L in some related extremality problem will also appear in the following).

Thus a very natural question appears from the very beginning: "How is the extremal solution ϕ encoded in its moments of degree less than or equal to N," or equivalently, "Which is the structure of the sequence of moments of an extremal solution, knowing that it depends only on a finite initial segment of it." Similarly to the classical one-variable case, we have two powerful tools to answer this question: an exponential E of a Cauchy-type transform of the extremal set and a hyponormal operator T associated canonically with the same set. For the function E we expect a rigid behaviour (so that it depends only on a finite initial segment of its Taylor expansion at infinity), while for the operator T we expect a matricial structure which again depends only on a finite dimensional block of it.

Both these rather vague pictures are validated by the one-variable case (exposed in detail in [10]) and by the situation of quadrature domains (in two variables) appearing in [15, 16]. We do not expect simple answers to these finite determination questions. However, any progress in this direction will have applications beyond the original context (paper [7] is such an example).

The paper is organized as follows. Section 2 recalls the essential results of Krein which characterize the extremal solutions of the truncated L-problem of moments. In Section 3 we restrict to dimension two, where we investigate the associated hyponormal operators and we establish, via a rational cyclicity criterion, the existence of a general analytic model of them, valid for all extremal solutions discussed in Section 2. Another point of view is taken in Section 4, where the canonical exponential kernel of an extremal solution of the moment problem is coming into focus. After a brief review of the main results of the parallel paper [7] (which reveal analytic continuation properties of the kernel) we present an intrinsic characterization of all these kernels. As a consequence, we interpret the L-problem of moments as an interpolation problem of the Carathéodory–Fejér type, for a class of analytic functions defined in the bidisk.

2. CONVEXITY RESULTS

This section contains a review of known facts derived from the work of M. G. Krein and his school. Due to their generality, the results are presented in a slightly more general setting ($\mathbf{R}^n$ rather than $\mathbf{R}^2$ and an arbitrary compact support K for the frame of the moment problem). The basic monographs we refer to for details are [8] and [10].

The variable in $\mathbf{R}^n$ is denoted by $x=(x_1, ..., x_n)$; dx means the volume measure in $\mathbf{R}^n$. For a multi-index $\alpha=(\alpha_1, ..., \alpha_n)\in\mathbf{N}^n$ we denote $|\alpha|=\alpha_1+\cdots+\alpha_n$ and put as usual $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$.

Let K be a compact subset of $\mathbf{R}^n$; in order to avoid some minor complications we assume that $\operatorname{int}(K)\neq\emptyset$. Fix a positive integer N and a positive constant L. The *truncated L-problem of moments* supported by the set K consists in finding necessary and sufficient conditions for a sequence $a=(a_\alpha)_{|\alpha|\leqslant N}$ of real numbers to be represented as:

$$a_\alpha=\int_K x^\alpha\phi(x)\,dx \qquad (|\alpha|\leqslant N,\ \phi\in L^1(K, dx),\ 0\leqslant\phi\leqslant L). \tag{1}$$

Moreover, it is of interest to classify all solutions of this problem and to characterize the uniqueness cases; see [10, Chap. VII].

For a first part of this section we can adopt the normalization $L=1$.

Let us denote, for $L=1$, the set of all possible moment sequences as follows:

$$\Sigma=\left\{a(\phi)=(a_\alpha);\ a_\alpha=\int_K x^\alpha\phi(x)\,dx,\ |\alpha|\leqslant N,\ \phi\in L^1(K, dx),\ 0\leqslant\phi\leqslant 1\right\}. \tag{2}$$

Let $\mathbf{R}[x]$ be the space of polynomials in the variables x, with real coefficients. We put $\mathscr{P}_N=\mathscr{P}_N(\mathbf{R}^n)=\{p\in\mathbf{R}[x], \deg(p)\leqslant N\}$. With these data fixed, we denote $d=\dim(\mathscr{P}_N)$ and parametrize the coordinates in the space $\mathscr{P}_N\subset\mathbf{R}^d$ as follows: $y=(y_\alpha)_{|\alpha|\leqslant N}$.

It is clear that Σ is a compact convex subset of $\mathbf{R}^d$. Let a^0 be a boundary point of Σ and let $f(y)=\langle c, y\rangle+d$ be a supporting affine functional to Σ, at the point a^0. Let us represent the point a^0 as the moment sequence of the function ϕ_0: $a^0 = a(\phi_0)$. Then we have the following relations:

$$\langle c, a \rangle + d \leqslant 0 \qquad (a \in \Sigma)$$

and

$$\langle c, a^0 \rangle + d = 0.$$

By subtracting them and representing a as $a = a(\phi)$, we obtain

$$\int_K p^0(x)(\phi(x) - \phi_0(x))\, dx \leqslant 0 \qquad (\phi \in L^1(K), 0 \leqslant \phi \leqslant 1), \tag{3}$$

where $p^0(x) = \sum_{|\alpha| \leqslant N} c_\alpha x^\alpha$. But relation (3) is possible for all functions ϕ as above if and only if $p^0(x) > 0$ implies $\phi_0(x) = 1$ and $p^0(x) < 0$ implies $\phi_0(x) = 0$. Since the set of zeroes of a nontrivial polynomial has null volume measure, these latter implications determine a unique element $\phi_0 \in L^1$. In other terms, we have proved that $\phi_0 = \chi_{\{p^0 > 0\}}$ a.e., where χ_S is the charactersistic function of the set S.

In fact, the above argument can easily be reversed, and we can state the following conclusion.

Proposition 2.1. *A point $(a_\alpha)_{|\alpha| \leqslant N}$ belongs to the boundary of the set Σ of all moments if and only if there is a nontrivial polynomial $p(x)$ of degree less than or equal to N, and with the property*:

$$a_\alpha = \int_{K \cap \{p > 0\}} x^\alpha\, dx \qquad (|\alpha| \leqslant N).$$

Above we have denoted in short by $\{p > 0\}$ the set of those points x which satisfy $p(x) > 0$. Since we have assumed that the compact set K possesses interior points, it is immediate to remark that $a(\phi) \in \mathrm{int}(\Sigma)$ for all functions ϕ satisfying $0 < \phi < 1$.

Following the monograph [10] we slightly change now the point of view. Since we assume the supporting compact set to be given, after a translation and homothety the problem (1) is equivalent to

$$2a_\alpha - L \int_K x^\alpha dx = \int_K x^\alpha (2\phi(x) - L)\, dx \qquad (|\alpha| \leqslant N),$$

where the new unknown function $2\phi - L$ satisfies $-L \leqslant 2\phi - L \leqslant L$. Modulo this transformation we consider henceforth the following moment problem:

$$a_\alpha = \int_K x^\alpha \phi(x)\, dx \qquad (|\alpha| \leqslant N, \phi \in L^1(K), -L \leqslant \phi \leqslant L). \tag{4}$$

Let us denote $a = (a_\alpha)_{|\alpha| \leqslant N}$ to be the sequence of virtual moments.

Because we have assumed $\operatorname{int}(K) \neq \emptyset$, the monomials $(x^\alpha)_{|\alpha| \leqslant N}$ are linearly independently regarded as functions of $x \in K$. Thus, for L large enough the problem (4) always admits a solution ψ.

Let us consider the embedding $\mathscr{P}_N(\mathbf{R}^n) \subset L^1_{\mathbf{R}}(K, dx)$ and consider the linear functional

$$l_a \colon \mathscr{P}_N \to \mathbf{R}, \qquad l_a(x^\alpha) = a_\alpha \qquad (|\alpha| \leqslant N).$$

By virtue of Riesz Theorem, any continuous extension of l_a to $L^1_{\mathbf{R}}(K)$ is represented by a function $\phi \in L^\infty_{\mathbf{R}}(K)$. Hence ϕ is a solution of problem (4) and we have:

$$l_a(p) \leqslant \|p\|_{1, K} \|\phi\|_{\infty, K} \qquad (p \in \mathscr{P}_N). \tag{5}$$

Then it remains to remark that the converse implication is given by the Hahn–Banach Theorem. Moreover, the familiar analysis of the equality case in (5) is also relevant for us. Summing up, we can state the next proposition.

PROPOSITION 2.2. (a) *Problem* (4) *admits a solution if and only if* $L \geqslant \sup\{(l_a(p)/\|p\|_{1,K});\, p \in \mathscr{P}_N \setminus \{0\}\}$.

(b) *If* $L = \sup\{(l_a(p)/\|p\|_{1,K});\, p \in \mathscr{P}_N \setminus \{0\}\}$, *then the solution is unique and it coincides with the function* $L \operatorname{sgn}(p_0)$, *where* p_0 *is a polynomial of degree less than or equal to N.*

For details of the proof of Proposition 2.2 and further comments see [10, Section IX.1-2]. In other terms, we can state as a consequence the following result.

COROLLARY 2.3. *The function* $\phi \in L^\infty_{\mathbf{R}}(K)$ *is uniquely determined in the ball* $\{\psi \in L^\infty_{\mathbf{R}}(K);\, \|\psi\|_{\infty,K} \leqslant \|\phi\|_{\infty,K}\}$ *by its finite sequence of moments* $a = a(\phi)$ *if and only if*

$$\phi = \|\phi\|_{\infty, K} \operatorname{sgn}(p),$$

where $p \in \mathscr{P}_N \setminus \{0\}$.

Returning now to problem (1), with constant $L=1$, we let the degree N change from a fixed value N to $N+1$. Let Σ_N and Σ_{N+1} denote the corresponding sets of moment sequences; let $\pi\colon \Sigma_{N+1} \to \Sigma_N$ be the canonical projection:

$$\pi((a_\alpha)_{|\alpha| \leqslant N+1}) = (a_\alpha)_{|\alpha| \leqslant N}.$$

Since the sets Σ_N, Σ_{N+1} are convex, the fibre $\pi^{-1}a$ of an element $a \in \Sigma_N$ is a convex subset of a linear variety which intersects $\partial\Sigma_{N+1}$. Thus, we can state the following result.

THEOREM 2.3. *Let K be a fixed compact subset of $\mathbf{R}^n$ with interior points. The moment problem* (1) *has a solution for $L=1$ if and only if there exists a poynomial P of degree $N+1$ such that*:

$$a_\alpha = \int_{K \cap \{P>0\}} x^\alpha \, dx \qquad (|\alpha| \leqslant N).$$

Contrary to the case $n=1$, for $n>1$ there is no constructive way of finding the polynomial P from the finite moment data $(a_\alpha)_{|\alpha| \leqslant N}$.

3. EXTREMAL HYPONORMAL OPERATORS

From now on we assume the dimension of the underlying space to be two ($n=2$). Only in this dimension the L-problem of moments can be interpreted as an inverse spectral problem for a class of hyponormal operators. The benefits of this relationship have begun to show up in the previous papers [15] and [16]. In the rest of the present paper we continue to exploit this operator theory interpretation of the L-problem, with the future prospect (not yet achieved) of understanding the structure of extremal sequences of moments. We use the lecture notes [12] as a general reference to the theory of hyponormal operators.

Let ϕ be a measurable function with compact support in the complex plane which satisfies $0 \leqslant \phi \leqslant 1$, a.e. Let T be the unique (up to unitary equivalence) hyponormal operator with rank-one self-commutator ($[T^*, T] = \xi \otimes \xi$) with the principal function equal to ϕ.

The two objects above are related by the following formula:

$$\langle (T^* - \bar{w})^{-1} \xi, (T^* - \bar{z})^{-1} \xi \rangle = 1 - \exp\left(\frac{-1}{\pi} \int_{\mathbf{C}} \frac{\phi(\zeta) \, dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})} \right),$$

which is valid for all points z, w in the reslovent set of the operator T. Actually, a separately continuous extension of the above formula to the whole $\mathbf{C}^2$ holds; see [2] and [12]. The importance of this formula lies in the fact that it relates, after taking the Taylor expansions at infinity, the moments of the function ϕ to the moments of the operator T with respect to the vector ξ.

We are interested in the following in the moments of the extremal solutions discussed in the preceding section. Therefore, assuming the supporting compact set K (introduced in Section 2) to be nice, for instance, a rectangle, we will study the above relationship only for characteristic functions $\phi = \chi_\Omega$, where Ω is a bounded domain of the complex plane, satisfying $\Omega = \operatorname{int}(\bar{\Omega})$ and possibly being defined by a single polynomial inequality. To simplify notation we put:

$$E_\Omega(z, \bar{w}) = \exp\left(\frac{-1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\right) \qquad (z, w \in \mathbf{C}\backslash\bar{\Omega}).$$

To each domain Ω as above we associate the unique irreducible hyponormal operator T satisfying $[T^*, T] = \xi \otimes \xi$ and having the principal function equal to χ_Ω. We simply call T *the hyponormal operator corresponding to* Ω. Recall that the spectrum of T is the closure of Ω, the essential spectrum is the boundary of Ω, and so on. See [12, Chapt. XI] for more details.

We know from previous work, [15], that in the case of a quadrature domain Ω, the corresponding operator T has a simple two block-diagonal matricial form. Although for more general domains a more complicated structure of T is expected, we prove first a general rational cyclicity result for all these operators. Some related results were obtained with different techniques in the papers [2, 3, and 14].

THEOREM 3.1. *Let T be a bounded hyponormal operator with rank-one self-commutator $[T^*, \,] = \xi \otimes \xi$ and with the principal function equal to the characteristic function of its spectrum. Then the vector ξ is rationally cyclic for T.*

Proof. Let H_1 be the rationally cyclic subspace for $T \in L(H)$ generated by the vector ξ. To be more specific, H_1 is the closure of the linear span of vectors of the form $r(T)\xi$, where r is a rational function with poles outside $\sigma(T)$. Let T_1 be the restriction of the operator T to H_1. It is well known that $\sigma(T_1) \subset \sigma(T)$ and, according to Berger–Shaw inequality, the hyponormal operator T_1 has a trace–class self-commutator; see [12, Corollary VI.1.5]. Let $g = \chi_{\sigma(T)}$ be the principal function of T and let g_1 denote the principal function of T_1. We know by the same technique that was used in the proof of the Berger–Shaw inequality that $0 \leqslant g_1 \leqslant 1$ a.e.; see [12, Proposition X.4.6].

By taking the orthogonal decomposition $H = H_1 \oplus H_1^{\perp}$ we obtain a matrix representation of the operator T:

$$T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}.$$

Since $\sigma(T_1) \subset \sigma(T)$, we obtain $\sigma(T_2) \subset \sigma(T)$. Moreover, the equation $[T^*, T] = \xi \otimes \xi$ reads on the above matrix components:

$$[T_1^*, T_1] - XX^* = \xi \otimes \xi \tag{6}$$

and

$$[T_2^*, T_2] + X^*X = 0. \tag{7}$$

Thus the operator X is Hilbert–Schmidt and consequently the operator T_2 is co-hyponormal with trace-class self-commutator. The principal function g_2 of T_2 satisfies $g_2 \leqslant 0$, a.e.

Because $\mathrm{Tr}[X^*, X] = 0$, relations (6) and (7) yield

$$\mathrm{Tr}[T_1^*, T_1] + \mathrm{Tr}[T_2^*, T_2] = \mathrm{Tr}(\xi \otimes \xi),$$

or equivalently, by the Helton–Howe formula,

$$\int_{\sigma(T)} (g_1 + g_2)\, dA = \int_{\sigma(T)} g\, dA.$$

On the other hand, we know that

$$g_2 \leqslant 0 \leqslant g - g_1$$

almost everywhere on $\sigma(T)$. Hence $g_2 = 0$ a.e. and the operator T_2 turns out to be normal. Therefore $X = 0$, and by the irreducibility of T we find that $T_2 = 0$.

This completes the proof of Theorem 3.1.

For sets K of finite perimeter, Theorem 3.1 was proved by methods of rational approximation theory by K. Clancey; see [2, Theorem 5]. For K equal to the closure of a domain bounded by finitely many smooth and nonintersecting real analytic curves, J. Pincus, D. Xia, and J. Xia have shown a stronger result, namely that the coresponding operator T is similar to the multiplication by z on the Hardy space of the boundary of K; see [14, Theorem 5].

Even in the case of an arbitrary compact set K, Theorem 3.1 has interesting applications. We confine ourselves here to discussing only one of them, but first we need some notation and terminology.

Let E_K be the exponential kernel corresponding to the set K and let $\mathscr{H}$ denote the Hilbert space completion of the space of test functions $\mathscr{D}(\mathbf{C})$ with respect to the seminorm:

$$\|\phi\|^2 = \frac{-1}{\pi^2} \int_{\mathbf{C}\times\mathbf{C}} E_K(w, \bar{z})\, \partial_z \phi(z)\, \partial_{\bar{w}} \overline{\phi(w)}\, dA(z)\, dA(w).$$

Then we know from previous work that the Hilbert space H is isometrically isomorphic to the weighted Sobolev type space $\mathscr{H}$, that in this isomorphism the operator T^* corresponds to the multiplication by $\bar{z}$, and the vector ξ corresponds to a smooth function χ with compact support in $\mathbf{C}$ which is equal to 1 in a neighbourhood of K; see [12, Proposition XI.3.1]. Let us remark that

$$-\chi_K = \lim_{w\to\infty} (\bar{w}\, \partial_{\bar{z}} E_K(\bar{w}, z)) = \partial_{\bar{z}} \langle \xi, (T^* - \bar{z})^{-1} \xi \rangle,$$

where the derivatives are taken in the sense of distributions. Let r be a rational function with poles outside $\sigma(T)$ and let $\phi \in \mathscr{D}(\mathbf{C})$.

The scalar product of the space $\mathscr{H}$ does not distinguish functions which vanish on a neighbourhood of K. Therefore we make an abuse of notation and regard the elements of $\mathscr{H}$ as limits of germs of functions defined on neighbourhoods of K.

Let us recall that

$$\lim_{w\to\infty} (\bar{w}(E_K(\bar{w}, z) - 1)) = \langle \xi, (T^* - \bar{z})^{-1} \xi \rangle = \frac{1}{\pi} \int_K \frac{dA(\zeta)}{\zeta - z},$$

see [2] or [12, Chapt. XI].

With these preparations, by an iterated integration and Stokes theorem (for the integral in w) we obtain

$$\begin{aligned} \langle \chi, (r(T))^* \phi \rangle_{\mathscr{H}} &= \frac{-1}{\pi^2} \langle \partial_z\, \partial_{\bar{w}} E_K(w, \bar{z}), \chi(w)\, r(z)\, \overline{\phi(z)} \rangle \\ &= \frac{1}{\pi^2} \left\langle \int_{\mathbf{C}} (T^* - \bar{w})^{-1}\, \xi\, \partial_w \chi(w)\, dA(w), \right. \\ &\qquad \left. \int_{\mathbf{C}} (T^* - \bar{z})^{-1}\, \xi\, \partial_z (\phi(z)\, \overline{r(z)})\, dA(z) \right\rangle. \end{aligned}$$

We can assume that the test function has support disjoint of the poles of the rational function r. Let ω be an open set with smooth boundary which contains $\bar{\Omega}$ and such that $\chi|\omega = 1$. Then the first integral above becomes:

$$\int_{\mathbf{C}} (T^* - \bar{w})^{-1} \xi \, \partial_w \chi(w) \, dA(w) = \frac{1}{2i} \int_{\mathbf{C} \setminus \bar{\omega}} (T^* - \bar{w})^{-1} \xi \, \partial_w \chi(w) \, d\bar{w} \wedge dw$$

$$= \frac{-1}{2i} \int_{\mathbf{C} \setminus \bar{\omega}} \partial_w [(T^* - \bar{w})^{-1} \xi \chi(w) \, d\bar{w}]$$

$$= \frac{1}{2i} \int_{\partial\omega} (T^* - \bar{w})^{-1} \xi \, d\bar{w} = \pi\xi.$$

Therefore

$$\langle \chi, r(T)^* \phi \rangle_{\mathscr{H}} = \pi^{-1} \int_{\mathbf{C}} \langle \xi, (T^* - \bar{z})^{-1} \xi \rangle \, \partial_{\bar{z}} [\overline{\phi(z)} \, r(z)] \, dA(z)$$

$$= \pi^{-1} \int_K \phi(z) \, \overline{r(z)} \, dA(z).$$

In conclusion we can state the following result.

Proposition 3.2. *Let, with the preceding notation, $\phi \in \mathscr{D}(\mathbf{C})$ and r be a rational function with poles outside the compact set K. Then the following formula holds*:

$$\pi \langle \phi, r(T) \chi \rangle_{\mathscr{H}} = \int_K \phi \bar{r} \, dA. \tag{8}$$

This is a generalization of the basic quadrature formula which was discussed in [16], namely:

$$\pi \langle \xi, r(T) \xi \rangle \mathscr{H} = \pi \langle \chi, r(T) \chi \rangle_{\mathscr{H}} = \int_K \bar{r} \, dA.$$

Let P denote the orthogonal projection of $L^2(K)$ onto the closure $R^2(K)$ of rational functions with poles off K in $L^2(K)$, all with respect to the area measure. Since the vectors $r(T)\xi$ form a dense subset of H (in virtue of Theorem 3.1), formula (8) shows that the linear densely defined map

$$F: R^2(K) \to \mathscr{H}, \qquad F(r) = r(T)\xi,$$

has as its adjoint, up to the constant π, also a densely defined map

$$P: \mathscr{H} \to R^2(K).$$

Note that these two maps are not likely to be bounded, and hence extendable to the whole spaces, as for instance the case of the unilateral shift shows. However, a functional model up to a one-sided quasi-similarity for these extremal operators can be obtained as follows.

Below we denote by $H^p(\Omega)$, $H^p_0(\Omega)$ the L^2-Sobolev spaces of the domain Ω of order p. The multiplication operator with the variable $\bar{z}$ is denoted by $M_{\bar{z}}$.

THEOREM 3.3. *Let Ω be a bounded domain of the complex plane, with piece-wise smooth boundary and let $T_\Omega \in L(H)$ be the irreducible hyponormal operator with rank-one self-commutator and principal function equal to the characteristic function of $\bar{\Omega}$.*

Then there exists a linear bounded operator $X: H^1(\Omega)/\partial H^2_0(\Omega) \to H$ with null kernel and dense range, and which is such that $XM_{\bar{z}} = T^*_\Omega X$.

Proof. The norm of a test function $\phi \in \mathcal{D}(\mathbf{C})$ in the space $\mathcal{H}$ can be given by the formula (see [12, Proposition XI.3.1]):

$$\|\phi\|^2_{\mathcal{H}} = \left\| \int_{\mathbf{C}} (T^* - \bar{z})^{-1}\, \partial\phi(z)\, dA(z) \right\|^2.$$

By using Stokes theorem and the fact that the boundary of Ω is piecewise smooth we obtain:

$$\int_{\mathbf{C}} (T^* - \bar{z})^{-1}\, \partial\phi(z)\, dA(z)$$

$$= \int_{\Omega} (T^* - \bar{z})^{-1}\, \partial\phi(z)\, dA(z) + \frac{1}{2i} \int_{\mathbf{C}\setminus\bar{\Omega}} \partial((T^* - \bar{z})^{-1}\, \phi(z))\, d\bar{z} \wedge dz$$

$$= \int_{\Omega} (T^* - \bar{z})^{-1}\, \partial\phi(z)\, dA(z) + \frac{1}{2i} \int_{\partial\Omega} (T^* - \bar{z})^{-1}\, \phi(z)\, d\bar{z}.$$

Since $\|(T^* - \bar{z})^{-1}\, \xi\| \leqslant 1$, $([T^*_\Omega, T_\Omega] = \xi \otimes \xi)$, there exists a constant C which depends only on the domain Ω, such that:

$$\|\phi\|_{\mathcal{H}} \leqslant C\, \|\phi\|_{H^1(\Omega)} \qquad (\phi \in \mathcal{D}(\mathbf{C})).$$

Therefore the map $J: H^1(\Omega) \to \mathcal{H}$ induced by the identity is continuous and has dense range by the very definition of the norm of the space $\mathcal{H}$.

According to Theorem 3.1 and relation (8), the element $J(\phi)$ is zero in $\mathcal{H}$ if and only if

$$\int_{\Omega} \phi \bar{r}\, dA = 0$$

for every rational function r with poles off $\bar{\Omega}$. But this is equivalent to the fact that the Cauchy transform

$$\psi(z) = \frac{-1}{\pi} \int_{\Omega} \frac{\phi(\zeta)\, dA(\zeta)}{\bar{\zeta} - \bar{z}}$$

is supported by the set $\bar{\Omega}$. Moreover, $\partial\psi = \phi$ in the sense of distributions and hence $\psi \in H_0^2(\Omega)$.

Since the operator T_{Ω}^* is similar to the multiplication $M_{\bar{z}}$ by $\bar{z}$ on $\mathscr{H}$, the map J induces the one-sided quasi-similarity X between the latter operator and $M_{\bar{z}}$, regarded as an operator on the quotient Hilbert space $H^1(\Omega)/\partial H_0^2(\Omega)$.

This finishes the proof of Theorem 3.3.

Let us remark that the above proof shows that in the statement of Theorem 3.3 we can replace the two classical Sobolev spaces with those defined only by norms of ∂-derivatives. In either case, it seems that the operator theory for the multiplier by $\bar{z}$ on the respective quotient space was not yet studied.

4. THE EXPONENTIAL KERNEL

Besides the associated hyponormal operator with rank-one self-commutator, the exponential kernel E_{Ω} of a bounded planar domain Ω is the only object which has proved to be useful in solving the L-problem of moments. Two specific properties of this kernel are of interest from the very beginning: the Taylor series at infinity (in both variables) of E_{Ω} determines the class $\chi_{\Omega} \in L^1(\mathbf{C})$, and moreover, the Taylor polynomial at infinity of degree N in both variables of E_{Ω} is in a simple (algebraic, nonlinear) bijection with the moments of degree less than or equal to N of the domain Ω.

A detailed investigation of the analytic properties of the exponential kernel is carried on in the paper [7]. We recall from there a single, central result.

THEOREM 4.1. *Let Ω be a bounded domain of the complex plane and let $E_{\Omega}(z, \bar{w})$, T be the associated exponential kernel and hyponormal operator with rank-one self-commutator. Let a be a fixed point in the boundary of Ω and let U be an open neighbourhood of a. The following assertions are equivalent*:

(a) *The Cauchy transform $\int_{\Omega} (dA(\zeta)/(\zeta - z))$ extends analytically from $U \backslash \bar{\Omega}$ to U.*

(b) *The kernel $E_{\Omega}(z, \bar{w})$ extends in both variables analytically/anti-analytically from $U \backslash \bar{\Omega}$ to U.*

(c) *The localized resolvent* $(T^* - \bar{z})^{-1}\,\xi$, *extends anti-analytically* (*as a vector valued function*) *from* $U \setminus \bar{\Omega}$ *to* U.

Moreover, in that case the boundary $U \cap \partial\Omega$ *is real analytic and* $E_\Omega(z, \bar{z}) = 0$ *for* $z \in U \cap \partial\Omega$.

The last part of Theorem 4.1 gives an alternative proof of a recent regularity result of Sakai [17] for boundaries admitting Schwarz functions. As a byproduct of the proof of Theorem 4.1, it is shown in [7] that the exponential kernel E_Ω extends analytically along paths, as long as the Schwarz function of an analytic arc of the boundary $\partial\Omega$ extends. See [18] for terminology and the significance of such phenomena.

Simple arguments show that the exponential kernel does not extend analytically across some singularities in the boundary of $\partial\Omega$ such as normal crossings or points of multiplicity higher than three.

Suppose now that Ω is an extremal domain for the L-problem, that is, $\Omega = K \cap \{P > 0\}$, where K is for instance a rectangle and P is a real polynomial of degree N. In that case the Schwarz function $S(z)$ of the boundary $\partial\Omega$ is an algebraic function in Ω which satisfies by definition the condition $S(z) = \bar{z}$, $z \in \partial\Omega$; see for details [4] and [18]. By putting together the above data we are faced with the following picture: the exponential kernel E_Ω is determined by its Taylor polynomial at infinity of degree N, and it is analytic/anti-analytic outside Ω and has the analytic extension behaviour of an algebraic function inside Ω (more precisely, the analytic extension in each variable has finitely many ramification points in Ω). Moreover, the analytic extension of E_Ω provides, when restricted on the diagonal, a canonical defining function of $\partial\Omega$. Thus, coming back to the question raised in the introduction we ask: "Knowing all these facts, is there a simple structure of the extremal exponential kernels?"

So far, only the case of quadrature domains proved to be successful in elucidating this question [15] or the image by an inversion of the exterior of a quadrature domain [7].

Trying to understand the exponential kernel of a domain we were led to an intrinsic characterization of it. The rest of this section is independent of the-other parts and it contains this result. The principle of obtaining such a characterization in terms of positive definite kernels is not new. It goes back to the work of de Branges on Hilbert spaces of analytic functions [5]; the same technique was exploited later by Pincus and Rovnyak [13], Carey and Pincus [1], and several other authors whose interests came in contact with determining or characteristic functions of various classes of operators. For another notable example see also Livšic [11]. In spite of several close similarities with the mentioned works, we believe that the details contained below are new.

To simplify notation we put $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, $\mathbf{D} = \{z \in \mathbf{C};\ |z| < 1\}$, and $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. For a measurable function $g: \mathbf{D} \to [0, 1]$ we denote:

$$E_g(z, \bar{w}) = \exp\left(\frac{-1}{\pi} \int_{\mathbf{D}} \frac{g(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\, dA(\zeta)\right) \qquad (|z|, |w| > 1).$$

Let us also remark that the above function extends analytically to a function

$$E_g: (\hat{\mathbf{C}} \setminus \bar{\mathbf{D}})^2 \to \mathbf{C}^*.$$

The question we address below is to find a set of minimal conditions which characterizes analytic function $E: (\hat{\mathbf{C}} \setminus \bar{\mathbf{D}})^2 \to \mathbf{C}^*$ to be of the form $E = E_g$ for a measurable function g as above.

One obvious condition is

$$E(\infty, \bar{w}) = E(z, \infty) = 1 \qquad (z, w \in \hat{\mathbf{C}} \setminus \bar{\mathbf{D}}). \tag{9}$$

In order to state the next condition we define a new kernel $F: (\hat{\mathbf{C}} \setminus \bar{\mathbf{D}})^4 \to \mathbf{C}$ by the formula:

$$F(z_1, \overline{z_2}; w_1, \overline{w_2}) = \frac{E(z_1, \overline{w_2})\, E(w_1, \overline{z_2}) - E(z_1, \overline{z_2})\, E(w_1, \overline{w_2})}{(w_1 - z_1)(\overline{w_2} - \overline{z_2})\, E(z_1, \overline{w_2})}. \tag{10}$$

Whenever we encounter an analytic function $h(z)$, the quotient

$$\frac{h(z) - h(w)}{z - w} \qquad (z \neq w)$$

is extended analytically across the diagonal $(z = w)$ by the value $h'(z)$. As a matter of terminology the inequality $K(z_1, \overline{z_2}; w_1, \overline{w_2}) \succ 0$ means that the kernel K is nonnegatively definite, that is,

$$\sum_{k, l = 1}^{N} K(s_k, \overline{t_k}; t_l, \overline{s_l})\, \lambda_k \overline{\lambda_l} \geqslant 0,$$

for every finite set of points $\{(s_k, t_k), 1 \leqslant k \leqslant N\}$ in the domain of K and every complex number λ_k, $1 \leqslant k \leqslant N$.

THEOREM 4.2. *Let $E: (\hat{\mathbf{C}} \setminus \bar{\mathbf{D}})^2 \to \mathbf{C}^*$ be an analytic function with the normalization property* (9) *and let F be the associated kernel* (10).

There is a measurable function $g: \mathbf{D} \to [0, 1]$ with the property that $E = E_g$ if and only if

$$F(z_1, \overline{z_2}; w_1, \overline{w_2}) \succ z_1 \overline{w_2} F(z_1, \overline{z_2}; w_1, \overline{w_2}) - (z_1 \overline{w_2}\, F(z_1, \overline{z_2}; w_1, \overline{w_2}))_{z_1=\infty} - (z_1 \overline{w_2}\, F(z_1, \overline{z_2}; w_1, \overline{w_2}))_{w_2=\infty} + (z_1 \overline{w_2}\, F(z_1, \overline{z_2}; w_1, \overline{w_2}))_{z_1=w_2=\infty} \succ 0. \tag{11}$$

Note that the second term in the above positivity condition is a second order difference at infinity of the function F.

Proof (Necessity). For this part of the proof we use the known factorization of the positive definite kernel $1 - E_g$ in terms of the associated hyponormal operator with rank-one self-commutator; see [12, 1].

Let g be a measurable function as in the statement and let T denote the irreducible hyponormal operator with rank-one self-commutator which has the principal function equal to g, almost everywhere. Let us denote (as before) $[T^*, T] = \xi \otimes \xi$ and let us recall the basic formula:

$$E_g(z, \bar{w}) = 1 - \langle (T^* - \bar{w})^{-1} \xi, (T^* - \bar{z})^{-1} \xi \rangle. \tag{12}$$

Note that we have tacitly made the normalization $\operatorname{supp}(g) \subset \overline{\mathbf{D}}$, hence $\sigma(T) \subset \overline{\mathbf{D}}$ and $\|T\| \leqslant 1$ (because T is a hyponormal operator; see [12, Corollary 3.1.4]).

Next we need a few elementary identities with resolvents, all stated for the current variables $u, v, z_1, ...$ outside the closed unit disk:

$$(T-u)^{-1}(T-v)^{-1} = \frac{(T-u)^{-1} - (T-v)^{-1}}{u-v} \tag{13}$$

and

$$(T^* - \bar{u})^{-1}(T-v)^{-1} = (T-v)^{-1}(T^* - \bar{u})^{-1} + (T^* - \bar{u})^{-1}(T-v)^{-1}(\xi \otimes \xi) \times (T-v)^{-1}(T^* - \bar{u})^{-1}. \tag{14}$$

In particular, from formula (14) we obtain:

$$\langle (T^* - \bar{u})^{-1}(T-v)^{-1} \xi, \xi \rangle = \langle (T-v)^{-1}(T^* - \bar{u})^{-1} \xi, \xi \rangle + \langle (T^* - \bar{u})^{-1}(T-v)^{-1} \xi, \xi \rangle \times \langle (T-v)^{-1}(T^* - \bar{u})^{-1} \xi, \xi \rangle,$$

or equivalently,

$$(1-\langle (T-v)^{-1}(T^*-\bar{u})^{-1}\xi,\xi\rangle)(1+\langle (T^*-\bar{u})^{-1}(T-v)^{-1}\xi,\xi\rangle)=1,$$

that is,

$$1+\langle (T^*-\bar{u})^{-1}(T-v)^{-1}\xi,\xi\rangle=\frac{1}{E_g(v,\bar{u})}. \tag{15}$$

We claim that

$$F(z_1,\overline{z_2};w_1,\overline{w_2})=\langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\\ (T-w_2)^{-1}(T^*-\overline{w_1})^{-1}\xi\rangle. \tag{16}$$

Indeed, according to these identities we obtain (denoting $E=E_g$):

$$\begin{aligned}
&\langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,(T-w_2)^{-1}(T^*-\overline{w_1})^{-1}\xi\rangle\\
&\quad=\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle\\
&\quad=\langle (T-w_1)^{-1}(T-z_1)^{-1}(T^*-\overline{w_2})^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle\\
&\qquad+\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T-z_1)^{-1}\xi,\xi\rangle\\
&\qquad\times\langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle\\
&\quad=\frac{\begin{bmatrix}\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle\\ -\langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle\end{bmatrix}}{\overline{w_2}-\overline{z_2}}\\
&\quad=\frac{\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle-\langle (T-w_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle}{(w_1-z_1)(\overline{w_2}-\overline{z_2})}\\
&\qquad+\frac{\langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle-\langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle}{(w_1-z_1)(\overline{w_2}-\overline{z_2})}\\
&\qquad+(\langle (T-w_1)^{-1}(T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle\\
&\qquad+\langle (T-w_1)^{-1}(T^*-\overline{w_2})^{-1}(T-z_1)^{-1}\xi,\xi\rangle\\
&\qquad\langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle)\\
&\qquad\times\frac{\langle (T-z_1)^{-1}(T^*-\overline{w_2})^{-1}\xi,\xi\rangle-\langle (T-z_1)^{-1}(T^*-\overline{z_2})^{-1}\xi,\xi\rangle}{\overline{w_2}-\overline{z_2}}
\end{aligned}$$

$$= \frac{E(w_1, \overline{z_2}) + E(z_1, \overline{w_2}) - E(z_1, \overline{z_2}) - E(w_1, \overline{w_2})}{(w_1 - z_1)(\overline{w_2} - \overline{z_2})}$$
$$+ \frac{\langle (T-w_1)^{-1}\, (T-z_1)^{-1}\, (T^* - \overline{w_2})^{-1}\, \xi, \xi \rangle (E(z_1, \overline{z_2}) - E(z_1, \overline{w_2}))}{E(z_1, \overline{w_2})(\overline{w_2} - \overline{z_2})}$$
$$= \frac{E(w_1, \overline{z_2}) + E(z_1, \overline{w_2}) - E(z_1, \overline{z_2}) - E(w_1, \overline{w_2})}{(w_1 - z_1)(\overline{w_2} - \overline{z_2})}$$
$$+ \frac{(E(z_1, \overline{w_2}) - E(w_1, \overline{w_2}))(E(z_1, \overline{z_2}) - E(z_1, \overline{w_2}))}{E(z_1, \overline{w_2})(w_1 - z_1)(\overline{w_2} - \overline{z_2})}$$
$$= \frac{E(z_1, \overline{w_2})\, E(w_1, \overline{z_2}) - E(w_1, \overline{w_2})\, E(z_1, \overline{z_2})}{E(z_1, \overline{w_2})(w_1 - z_1)(\overline{w_2} - \overline{z_2})}.$$

Thus relation (16) is verified. It remains to remark that:

$$T(T-z_1)^{-1}\, (T^* - \overline{z_2})^{-1}\, \xi$$
$$= (T^* - \overline{z_2})^{-1}\, \xi + z_1 (T - z_1)^{-1}\, (T^* - \overline{z_2})^{-1}\, \xi$$
$$= z_1 (T-z_1)^{-1}\, (T^* - \overline{z_2})^{-1}\, \xi - (z_1 (T-z_1)^{-1}\, (T^* - \overline{z_2})^{-1}\, \xi)_{z_1 = \infty}.$$

In conclusion, the positivity conditions (11) in the statement become:

$$0 \prec \langle T(T-z_1)^{-1}\, (T^* - \overline{z_2})^{-1}\, \xi,\, T(T-w_2)^{-1}\, (T^* - \overline{w_1})^{-1}\, \xi \rangle$$
$$\prec \langle (T-z_1)^{-1}\, (T^* - \overline{z_2})^{-1}\, \xi,\, T(T-w_2)^{-1}\, (T^* - \overline{w_1})^{-1}\, \xi \rangle.$$

Since T is a contraction these two positivity conditions are evidently true.

(Sufficiency) Let E be an analytic function which satisfies the normalization and positivity conditions in the statement. We want to prove that $E = E_g$, where $g\colon \mathbf{D} \to [0, 1]$ is a measurable function. This in turn is equivalent to finding a hyponormal operator T with rank-one self-commutator $[T^*, T] = \xi \otimes \xi$ which has g as its principal function and hence E as an associated determinantal function.

Since the kernel F was supposed to be nonnegatively definite, there exists a separable, complex Hilbert space H and an H-valued analytic function

$$\rho\colon (\hat{\mathbf{C}} \backslash \overline{\mathbf{D}})^2 \to H$$

such that

$$F(z_1, \overline{z_2}; w_1, \overline{w_2}) = \langle \rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1}) \rangle \ \ (z_j, w_j \in \hat{\mathbf{C}} \backslash \overline{\mathbf{D}};\ j = 1, 2). \quad (17)$$

In addition, we can assume without loss of generality that the image of the function ρ spans the whole Hilbert space H.

By assumption, $F(\infty, \overline{z_2}; w_1, \overline{w_2}) = F(z_1, \infty; w_1, \overline{w_2}) = 0$, therefore

$$\rho(\infty, \bar{z}) = \rho(z, \infty) = 0 \qquad (z \in \hat{\mathbf{C}} \backslash \overline{\mathbf{D}}).$$

In particular, both limits $\lim_{z_1 \to \infty} z_1 \rho(z_1, \overline{z_2})$ and $\lim_{z_2 \to \infty} \overline{z_2}\rho(z_1, \overline{z_2})$ exist.

We define a linear transformation on the range of ρ by the formula:

$$T\rho(z_1, \overline{z_2}) = z_1 \rho(z_1, \overline{z_2}) - (z_1 \rho(z_1, \overline{z_2}))_{z_1 = \infty}. \tag{18}$$

Let n be a positive integer and let us choose arbitrary elements $z_1(k), z_2(k) \in \hat{\mathbf{C}} \backslash \overline{\mathbf{D}}$, $\lambda_k \in \mathbf{C}$, $1 \leqslant k \leqslant n$. In view of condition (11) in the statement of Theorem 4.2 we have:

$$\left\| T \sum_{k=1}^{n} \lambda_k \rho(z_1(k), \overline{z_2(k)}) \right\| \leqslant \left\| \sum_{k=1}^{n} \lambda_k \rho(z_1(k), \overline{z_2(k)}) \right\|.$$

Therefore, the map T extends linearly to a contraction defined on the whole space H. We denote its extension by the same symbol T.

Our next aim is to prove the formula

$$T^* \rho(z_1, \overline{z_2}) = \overline{z_2}\rho(z_1, \overline{z_2}) - (E(z_1, \overline{z_2})(\overline{z_2}\rho(z_1, \overline{z_2})))_{z_2 = \infty}. \tag{19}$$

To this end we choose arbitrary points $z_1, z_2, w_1, w_2 \in \hat{\mathbf{C}} \backslash \overline{\mathbf{D}}$ and compute

$$\begin{aligned}
&\langle T^* \rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1}) \rangle - \langle \rho(z_1, \overline{z_2}), T\rho(w_2, \overline{w_1}) \rangle \\
&\quad = \langle \overline{z_2}\rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1}) \rangle - E(z_1, \overline{z_2})(\langle \overline{z_2}\rho(z_1, \overline{z_2}), \rho(w_1, \overline{w_1}) \rangle)_{z_2 = \infty} \\
&\qquad - \overline{w_2} \langle \rho(z_1, \overline{z_2}), \rho(w_2, \overline{w_1}) \rangle + (\langle \rho(z_1, \overline{z_2}), w_2 \rho(w_2, \overline{w_1}) \rangle)_{w_2 = \infty} \\
&\quad = (\overline{z_2} - \overline{w_2})\, F(z_1, \overline{z_2}; w_1, \overline{w_2}) - E(z_1, \overline{z_2})(\overline{z_2} F(z_1, z_2; w_1, \overline{w_2}))_{z_2 = \infty} \\
&\qquad + (\overline{w_2} F(z_1, \overline{z_2}; w_1, \overline{w_2}))_{w_2 = \infty} \\
&\quad = \frac{\left[\begin{array}{c} -E(z_1, \overline{w_2})\, E(w_1, \overline{z_2}) + E(z_1, \overline{z_2})\, E(w_1, \overline{w_2}) \\ + E(z_1, \overline{z_2})(E(z_1, \overline{w_2}) - E(w_1, \overline{w_2})) \end{array} \right]}{(w_1 - z_1)\, E(z_1, \overline{w_2})} \\
&\qquad + \frac{E(w_1, \overline{z_2}) - E(z_1, \overline{z_2})}{w_1 - z_1} = 0.
\end{aligned}$$

Thus formula (19) is verified.

Let us remark that, denoting

$$\xi = (z_1\, \overline{z_2}\, \rho(z_1, \overline{z_2}))_{z_1 = z_2 = \infty},$$

we have

$$(T^*-\overline{z_2})(T-z_1)\,\rho(z_1,\overline{z_2})=(T^*-\overline{z_2})(-z_1\rho(z_1,\overline{z_2}))_{z_1=\infty}$$
$$=(E(z_1,\overline{z_2})(z_1\overline{z_2}\,\rho(z_1,\overline{z_2}))_{z_2=\infty})_{z_1=\infty}=\xi$$

(because $E(\infty,\overline{z_2})=1$). Whence we find the formula:

$$\rho(z_1,\overline{z_2})=(T-z_1)^{-1}\,(T^*-\overline{z_2})^{-1}\,\xi \qquad (z_1,z_2\in\hat{\mathbf{C}}\backslash\overline{\mathbf{D}}). \tag{20}$$

Consequently, we obtain

$$[T^*,T]\,\rho(z_1,\overline{z_2})=[T^*-\overline{z_2},T-z_1](T-z_1)^{-1}\,(T^*-\overline{z_2})^{-1}\,\xi$$
$$=\xi-E(z_1,\overline{z_2})\,\xi$$
$$=(1-E(z_1,\overline{z_2}))\,\xi.$$

Therefore the operator T has rank-one self-commutator, and the vector ξ spans the range of $[T^*,T]$.

Finally, we return to formula (10) and remark that

$$1-E(z_1,\overline{z_2})=(w_1\overline{w_2}\,F(z_1,\overline{z_2};w_1,\overline{w_2}))_{w_1=w_2=\infty}=\langle\rho(z_1,\overline{z_2}),\xi\rangle,$$

so that

$$[T^*,T]\,\rho(z_1,\overline{z_2})=\langle\rho(z_1,\overline{z_2})\xi,\xi\rangle\xi.$$

This proves that $[T^*,T]=\xi\otimes\xi$.

In conclusion, $E=E_g$, where g is the principal function of the operator T. This finishes the proof of Theorem 4.2.

Remark 4.3. By changing the variables $u_j=(1/z_j)$, $v_j=(1/w_j)$, $j=1,2$, we can define the function

$$G(u_1,\overline{u_2};v_1,\overline{v_2})=\frac{F(z_1,\overline{z_2};w_1,\overline{w_2})}{z_1\overline{w_2}}$$

so that G is analytic in the polydisk $\mathbf{D}^4$.

For an analytic function $h(z)$, $z\in\mathbf{D}$, we define the difference of h at zero by

$$\Delta_z h(z)=\frac{h(z)-h(0)}{z}.$$

Then condition (11) becomes:

$$0\prec\Delta_{u_1}\Delta_{\overline{v_2}}G(u_1,\overline{u_2};v_1,\overline{v_2})\prec G(u_1,\overline{u_2};v_1,\overline{v_2}). \tag{21}$$

Let $\mathscr{G}$ denote the class of all analytic functions $G: \mathbf{D}^4 \to \mathbf{C}$ which satisfy the positivity conditions (27) and have the structure derived from fomula (10), where E is subject to the normalization (9).

Then the truncated, or full, L-problem of moments treated in the previous sections (and in the papers [15, 16]) is equivalent to the following interpolation problem for the class $\mathscr{G}$:

$$G \in \mathscr{G}$$

and

$$((\partial/\partial u_1)^m (\partial/\partial \overline{u_2})^n G)(0, 0; 0, 0) = b_{mn} \qquad (0 \leqslant m \leqslant m+n \leqslant N).$$

This is a two-dimensional variant of the classical Carathéodory–Fejér problem (see for instance [6]). On the basis of our previous results obtained for the L-problem of moments, we know for the above interpolation problem how to describe its solvability in positivity terms (in the case $N = \infty$), while for the corresponding truncated interpolation problem we know a description of all its extremal solutions (for N finite). In view of the bijection between the class of functions $\mathscr{G}$ and the measurable functions $g: \mathbf{D} \to [0, 1]$, the class $\mathscr{G}$ has a natural convex structure given by the free parameter g.

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REFERENCES

1. R. W. Carey and J. D. Pincus, An exponential formula for determining functions, *Indiana Univ. Math. J.* **23** (1974), 1031–1042.
2. K. Clancey, The Cauchy transform of the principal function associated with an nonnormal operator, *Indiana Univ. Math. J.* **34** (1985), 21–32.
3. K. Clancey, Hilbert space operators with one dimensional self-commutators, *J. Operator Theory* **13** (1985), 265–289.
4. P. Davis, "The Schwarz Function and Its Applications," Carus Math. Monographs, Vol. 17, Math. Assoc. of America, 1974.
5. L. de Branges, "Hilbert Spaces of Entire Functions," Prentice–Hall, New Jersey, 1968.
6. C. Foiaş and A. E. Frazho, "The Commutant Lifting Approach to Interpolation Problems," Birkhäuser-Verlag, Basel, 1990.
7. B. Gustafsson and M. Putinar, An exponential transform and regularity of free boundaries in two dimensions, *Ann. Sci. Norm. Sup. Pisa*, to appear.
8. S. Karlin and W. J. Studden, "Tchebycheff Systems, with Applications in Analysis and Statistics," Interscience, New York, 1966.

9. M. G. Krein, The ideas of P. L. Chebyshev and A. A. Markov in the theory of limiting values of integrals and their further developments, *Uspekhi Mat. Nauk.* **6** (1951), 3–120 [in Russian].
10. M. G. Krein and A. A. Nudelman, "Markov Moment Problem and Extremal Problems," Translations, American Math. Society, Vol. 50, Amer. Math. Soc., Providence, RI, 1977.
11. M. S. Livšic, "Commuting Nonselfadjoint Operators and Collective Motions of Systems," Lecture Notes in Math., Vol. 1272, pp. 4–38, Springer-Verlag, Berlin.
12. M. Martin and M. Putinar, "Lectures on Hyponormal Operators," Birkhäuser, Basel, 1989.
13. J. D. Pincus and J. Rovnyak, A representation for determining functions, *Proc. Amer. Math. Soc.* **22** (1969), 498–502.
14. J. D. Pincus, D. Xia, and J. Xia, An analytic model for operators with one-dimensional self-commutator, *Int. Equations Operator Theory* **7** (1984), 516–535.
15. M. Putinar, Extremal solutions of the two-dimensional L-problem of moments, *J. Funct. Anal.* **136** (1996), 331–364.
16. M. Putinar, Linear analysis of quadrature domains, *Ark. Mat.* **33** (1995), 357–376.
17. M. Sakai, Regularity of a boundary having a Schwarz function, *Acta Math.* **166** (1991), 263–297.
18. H. S. Shapiro, "The Schwarz Function and Its Generalization to Higher Dimensions," Univ. of Arkansas Lecture Notes in Math., Vol. 9, Wiley, New York, 1992.